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SPACETIME DUALITY AND $\frac{SU(n,1)}{SU(n)\otimes U(1)}$ COSETS OF ORBIFOLD COMPACTIFICATION

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ABSTRACT

The duality symmetry group of the cosets $\frac{SU(n,1)}{SU(n)\otimes U(1)}$, which describe the moduli space of a two-dimensional subspace of an orbifold model with $(n-1)$ complex Wilson lines moduli, is discussed. The full duality group and its explicit action on the moduli fields are derived.

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The derivation of a field theoretic low-energy effective action from string theory is a first step in attempting to relate string theory to the supersymmetric standard model or grand unified theories. In particular, such a field theory should have implications to fundamental questions in particle physics and cosmology, such as the study of gauge coupling unification scale, the masses of quarks and leptons and stringy inspired inflationary scenarios.

Orbifold compactified heterotic string theories are of great phenomenological relevance. They constitute a large class of string vacua where the interactions can be computed explicitly using the underlying world-sheet conformal field theory [6]. The low-energy action is an $N = 1$ supergravity coupled to Yang-Mills and matter fields and their supersymmetric partners. With only terms with up to two derivatives in the bosonic fields, the theory is described in terms of three functions- the Kähler potential K encoding the kinetic terms for the massless fields, the superpotential W containing the Yukawa couplings and the gauge $f-$function whose real part, at the tree level, determines the gauge couplings [4]. The functions K and W appear in the Lagrangian of the theory through the combination

$$\mathcal{G} = K + log|W|^2. \tag{1}$$

The orbifold models posses a set of continuous parameters, the toroidal moduli, parametrizing the size and shape of the orbifold. The vacuum expectation values of the moduli fields represent marginal deformations of the underlying conformal field theory of the orbifold [5]. The toroidal moduli fields belong to the untwisted sector of the orbifold and enter the space-time $N = 1$ supersymmetric four-dimensional Lagrangian as chiral fields with flat potentials to all orders in perturbation theory. In addition to toroidal moduli, the untwisted sector of the heterotic string theory compactified on orbifolds may also contain Wilson lines moduli [11]. These additional moduli exist in orbifold models where the twist defining the orbifold is realized on the $E_8 \times E_8$ root lattice by a rotation [11]. Wilson line moduli are phenomenologicaly interesting because they lower the rank of the gauge group and thus leading to more realistic models. The moduli of the compactification on a d-dimensional torus $\mathbf{T}^d = \frac{\mathbf{R}^d}{\Lambda}$, where Λ is a d-dimensional lattice, are encoded in the metric G_{ij} which is the lattice metric of Λ, an antisymmetric tensor B_{ij} and Wilson lines ${A^I}_i$, where I is an $E_8 \times E_8$ gauge lattice index and i is an internal lattice

index. The moduli space of toroidal compactification [10] is given (locally) by the coset space $\frac{SO(d+16,d)}{SO(d+16,d)\otimes SO(d)}$. Toroidal compactifications lead to low-energy models with $N=4$ supersymmetry and gauge groups of rank $d+16$. Six-dimensional orbifolds [1,2] are obtained by identifying the points of the six-torus $\mathbf{T}^6$ under a cyclic group $Z_N=\{\theta^j, j=0,\cdots,N-1\}$. In order to obtain consistent space-time supersymmetric theories, the twist should belong to $SU(3)$ but not $SU(2)$ [1, 3]. Furthermore, to reduce the rank of the gauge group, continuous Wilson line moduli must be introduced, this can be achieved by allowing the orbifold twist to act on the gauge sector of the theory as an automorphism of the $E_8\times E_8$ root lattice. It can be demonstrated [8, 12] that the moduli spaces of orbifolds depend entirely on the eigenvalues of the twist and their multiplicities.

The moduli space of the orbifold are parametrized by the T moduli corresponding to the Kähler deformations and the U moduli which correspond to the deformations of the complex structure. For each U modulus, the corresponding moduli space is described by the coset $\left[\frac{SU(1,1)}{U(1)}\right]$, and apart from the $\mathbf{Z}_3$ orbifold*, the T moduli spaces for all symmetric orbifolds yielding $N=1$ space-time supersymmetry are given by the special Kähler manifolds [17]

$$\mathbf{SK}(n+1)=\frac{SU(1,1)}{U(1)}\otimes\frac{SO(n,2)}{SO(n)\otimes SO(2)}, \qquad n=2,4.$$

This structure can be derived using the Ward-identities of the underlying world-sheet $(2,2)$ superconformal algebra [7].

In the presence of Wilson lines moduli, the twist has more eigenvalues due to the enlarged action of the twist on the $E_8\times E_8$ lattice and the moduli spaces are given by [8]

$$\bigotimes_{i=1}^{n}\frac{SU(m_i,n_i)}{SU(m_i)\otimes SU(n_i)\otimes U(1)}\otimes\frac{SO(p,q)}{SO(p)\otimes SO(q)}, \tag{2}$$

where $m_i(n_i)$ and p (q) are the multiplicities of the complex and -1 eigenvalues of the twist on the left (right) moving sector, respectively.

$\star$ The moduli space of the Z_3 orbifold is $\frac{SU(3,3)}{SU(3)\otimes SU(3)\otimes U(1)}$, which is also special Kähler.

A peculiar but phenomenologicaly interesting feature of string compactifications is that the physical parameters of the low-energy effective theory are moduli dependent. Also, the theory has the target space duality symmetry, the T duality, which holds to all orders in perturbation theory (see [13] for a review). T-duality symmetries consist of discrete automorphisms of the moduli space which leave the underlying conformal field theory invariant. It is widely believed that non-perturbative effects in string theory provide the mechanism for a range of unsolved problems, namely, supersymmetry breaking, lifting the vacuum degeneracy in perturbative string theory and generating a non-trivial potential for the dilaton field. Also, non-perturbative potentials should have bearing on the questions of cosmological inflation and the cosmological constant. Presently, string theory is perturbative in its formulation and physical principles by which non-perturbative physics can be derived are not available. In this sense, duality symmetry play a major role, if assumed to hold non-perturbatively, it puts strong constraints on the form of any possible non-perturbative superpotential in the four dimensional low-energy effective action [16].

It is our purpose here to derive the full duality symmetry and its action on the moduli parameterizing the cosets $\frac{SU(n,1)}{SU(n)\otimes U(1)}$. These moduli spaces appear in orbifold models where Wilson lines moduli are present. So far only the action of a subgroup of the duality symmetry on the moduli representing these cosets has been discussed [8]. As a warm up exercise, we start by reviewing the duality symmetries for the coset $\frac{SO(2,2)}{SO(2)\otimes SO(2)}$, describing the moduli space of a two-dimensional toroidal or $\mathbf{Z}_2$ orbifold without Wilson lines. Then, duality symmetry for two-dimensional $\mathbf{Z}_N$ orbifold or a two dimensional subspace of a factorizable six-dimensional orbifold with a Z_N twist ($N \neq 2$), whose moduli space is given by $\frac{SU(1,1)}{SU(1)}$, is determined in terms of both $SL(2)$ and $SU(1,1)$ groups. In terms of $SL(2)$, the duality group is given by all those elements with integer values, *i.e.*, $SL(2,\mathbf{Z})$. However, in terms of $SU(1,1)$, it will be demonstrated that the duality group is not $SU(1,1,\mathbf{Z})$ but a subgroup whose elements depend on the particular twist defining the orbifold. These calculations are then extended to the cosets $\frac{SU(n,1)}{SU(n)\otimes U(1)}$, representing the moduli spaces of two-dimensional subspaces of orbifold compactification where continuous Wilson lines are present and where the twist has a complex eigenvalue. The full duality group and its action on the moduli fields are derived. Finally, we comment on the duality transformations of the basic physical parameters defining the low-energy effective action.

Duality symmetry is a discrete symmetry acting on the the moduli space of the underlying conformal field theory. The action on the moduli is such that the underlying conformal field theory is invariant. The vertex operators of the underlying conformal theory depend both on the moduli and a set of quantum numbers, the winding, momenta and gauge quantum numbers, this implies that duality symmetry has a non-trivial action on the quantum numbers in order to have a duality invariant spectrum. We will discuss the duality symmetry of a two-dimensional torus and its corresponding $\mathbf{Z}_N$ orbifolds.

The two-dimensional toroidal compactification is described by four real parameters represented by the independent components of the antisymmetric tensor B_{ij}, and the lattice metric G_{ij}. The vertex operators of the underlying conformal field theory have the following spins and scaling dimensions (ignoring the oscillators contribution)

$$H=\frac{1}{2}(P_L^tG^{-1}P_L+P_R^tG^{-1}P_R), \qquad S=\frac{1}{2}(P_L^tG^{-1}P_L-P_R^tG^{-1}P_R), \tag{3}$$

where the left and right momenta are given by

$$P_L=\frac{p}{2}+(G-B)w, \qquad P_R=\frac{p}{2}-(G+B)w, \tag{4}$$

here the index t denotes the transpose, w and p, the windings and momenta, respectively, are two-dimensional integer valued vectors taking values on the two dimensional lattice Λ of the torus and its dual Λ^* and G and B are 2×2 matrices representing the background metric and antisymmetric tensor.

To identify the symmetries of the spectrum, it is more convenient to write H and S in a matrix form [14]

$$H=\frac{1}{2}u^t\Xi u, \qquad S=\frac{1}{2}u^t\eta u, \tag{5}$$

where

$$\Xi=\begin{pmatrix} 2(G-B)G^{-1}(G+B) & BG^{-1} \\ -G^{-1}B & \frac{1}{2}G^{-1}\end{pmatrix}, \qquad \eta=\begin{pmatrix}\mathbf{0} & \mathbf{1}\\ \mathbf{1} & \mathbf{0}\end{pmatrix}, \qquad u=\begin{pmatrix} n_1\\ n_2\\ m_1\\ m_2\end{pmatrix}. \tag{6}$$

In terms of the complex moduli defined as,

$$T = T_1 + iT_2 = 2(\sqrt{\det G} - ib), \qquad U = U_1 + iU_2 = \frac{1}{G_{11}}(\sqrt{\det G} - iG_{12}), \tag{7}$$

Ξ is expressed by

$$\Xi = \frac{1}{T_1 U_1} \begin{pmatrix} |T|^2 & -|T|^2 U_2 & -T_2 U_2 & -T_2 \\ -|T|^2 U_2 & |T|^2 |U|^2 & T_2 |U|^2 & T_2 U_2 \\ -T_2 U_2 & T_2 |U|^2 & |U|^2 & U_2 \\ -T_2 & T_2 U_2 & U_2 & 1 \end{pmatrix}, \tag{8}$$

and

$$H = |\mathbf{P}_R|^2 + S = 2\frac{|m_2 - im_1 U + in_1 T - n_2 UT|^2}{(T+\bar{T})(U+\bar{U})} + S. \tag{9}$$

The discrete target space duality symmetries are defined by all integer-valued linear transformations of the quantum numbers which leave the spectrum invariant, this means that the duality transformations must leave both S and $|\mathbf{P}_R|^2$ invariant. Denote such a transformation by Ω with an action on the quantum numbers defined by

$$u \to \Omega^{-1} u. \tag{10}$$

It is then clear that for S to be invariant, Ω must satisfy

$$\Omega^t \eta \Omega = \eta. \tag{11}$$

Also the requirement of the invariance of H under the action of Ω, defines the duality transformation of the moduli fields. This is given by

$$\Xi \to \Omega^t \Xi \Omega. \tag{12}$$

In terms of the T and U parametrization of the moduli space, the duality symmetry can be

represented by $SL(2,\mathbf{Z})_T \otimes SL(2,\mathbf{Z})_U \otimes \mathbf{Z}_2^{(1)} \otimes \mathbf{Z}_2^{(2)}$ which act on the moduli as

$$\begin{aligned}
SL(2,\mathbf{Z})_T: &\quad T \to \frac{aT - ib}{icT + d}, \quad U \to U \qquad ad - bc = 1, \\
SL(2,\mathbf{Z})_U: &\quad U \to \frac{a'U - ib'}{ic'U + d'}, \quad T \to T \qquad a'd' - b'c' = 1, \\
\mathbf{Z}_2^{(1)}: &\quad T \leftrightarrow U, \\
\mathbf{Z}_2^{(2)}: &\quad T \leftrightarrow \bar{T}, \quad U \leftrightarrow \bar{U}.
\end{aligned} \tag{13}$$

The $SL(2,\mathbf{Z})_T$ and $SL(2,\mathbf{Z})_U$ action on the quantum numbers is represented by [15]

$$\Omega_T = \begin{pmatrix} a & 0 & 0 & c \\ 0 & a & -c & 0 \\ 0 & -b & d & 0 \\ b & 0 & 0 & d \end{pmatrix}, \qquad \Omega_U = \begin{pmatrix} d' & b' & 0 & 0 \\ c' & a' & 0 & 0 \\ 0 & 0 & a' & -c' \\ 0 & 0 & -b' & d' \end{pmatrix}. \tag{14}$$

The generalization of the above results to the orbifold is straightforward. Define the action of the twist on the quantum numbers by

$$u \longrightarrow u' = \Theta u, \qquad \Theta = \begin{pmatrix} Q & 0 \\ 0 & (Q^t)^{(-1)} \end{pmatrix} \qquad \Theta^N = 1, \tag{15}$$

here Q is an integer-valued matrix and N is the order of the twist. For the twist to act as a lattice automorphism, the background fields must satisfy the conditions

$$\Theta^t \Xi \Theta = \Xi, \quad \Rightarrow \qquad Q^t G Q = G, \qquad Q^t B Q = B. \tag{16}$$

Clearly, the conditions (16) imply that the orbifold has less moduli than its corresponding torus. The duality symmetry of the orbifold are those of the corresponding torus satisfying the additional condition $\Theta\Omega = \Omega\Theta^k$ with $1 \leq k \leq N$ [8, 14]. It is obvious that a two-dimensional $\mathbf{Z}_2$ has the same duality symmetry as that of the corresponding torus. However, for a $\mathbf{Z}_N$ twist, with $N \neq 2$, the U modulus is frozen to a constant complex value, and the duality symmetry is for two-dimensional $\mathbf{Z}_N$ orbifolds is $SL(2,\mathbf{Z})_T$. This can be easily from eq. (9) after setting U to a constant value U_0.

In order to identify the duality symmetry group of the coset $\frac{SU(1,1)}{U(1)}$ in terms of the group $SU(1,1)$, a complex basis u_c for the quantum numbers is required in which the spin is given by an $SU(1,1)$ quadratic form. A way to get this new basis of the quantum numbers is to use a different parametrization of the moduli space. If one writes $U_0 = u_1 + iu_2$, and perform the change of variables $t = \frac{1-T'}{1+T'}$, with $T' = \frac{T}{2u_1}$, then from (9) we obtain

$$|\mathbf{P}_R|^2 = 2\frac{|m_c - n_c t|^2}{(1-t\bar{t})} = u_c^\dagger \Xi_c u_c, \qquad S = u_c^\dagger L u_c \tag{17}$$

where⋆

$$\begin{aligned}
&L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u_c = \begin{pmatrix} n_c \\ m_c \end{pmatrix}, \quad \Xi_c = \frac{2}{1-t\bar{t}}\begin{pmatrix} t\bar{t} & -\bar{t} \\ -t & 1 \end{pmatrix}, \\
&m_c = \frac{1}{2\sqrt{2}u_1}(m_2 - im_1U_0 + 2in_1u_1 - 2n_2u_1U_0), \\
&n_c = \frac{1}{2\sqrt{2}u_1}(-m_2 + im_1U_0 + 2in_1u_1 - 2n_2u_1U_0).
\end{aligned} \tag{18}$$

An element of $SU(1,1)$ represented by Ω_c, is a 2×2 complex matrix with unit determinant satisfying

$$\Omega_c^\dagger L \Omega_c = L, \qquad \Omega_c = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}. \tag{19}$$

The condition in (19) implies

$$\bar{z}_1 = z_4, \qquad \bar{z}_2 = z_3. \tag{20}$$

If we define the action of Ω_c on the quantum numbers by

$$u_c \to \Omega_c^{-1} u_c, \tag{21}$$

then clearly this action leaves S invariant. For $|P_R|^2$ to remain invariant under the transformation

⋆ expressions for m_c and n_c have been obtained in [9] for the two dimensional $\mathbf{Z}_3$ orbifold case using a different approach. The expressions in (20) are valid for all orbifolds.

(21), the moduli field should transform as follows

$$\Xi_c \rightarrow \Omega_c^\dagger \Xi_c \Omega_c. \tag{22}$$

From (22) one can extract the duality transformation for the moduli t. This is given by

$$t \rightarrow \frac{z_1 t - z_3}{z_4 - z_2 t} = \frac{z_1 t - \bar{z}_2}{\bar{z}_1 - z_2 t}; \quad |z_1|^2 - |z_2|^2 = 1. \tag{23}$$

The elements of $SU(1,1)$ transformations can be expressed in terms of those of $SL(2, \mathbf{Z})$ by

$$\begin{aligned} z_1 = \bar{z}_4 = & \frac{1}{2}(a+d) + \frac{i}{2}(\frac{b}{2u_1} - 2u_1 c), \\ z_2 = \bar{z}_3 = & \frac{1}{2}(a-d) + \frac{i}{2}(\frac{b}{2u_1} + 2u_1 c). \end{aligned} \tag{24}$$

We stress here that the duality group is not $SU(1,1,\mathbf{Z})$ but a subgroup of $SU(1,1)$ whose elements depend on the particular orbifold (u_1 is different for different orbifolds).

We now turn to discuss the full duality symmetry of the moduli space of a two-dimensional subspace of a factorizable six-dimensional orbifold, where Wilson lines are present and where the twist is given by Z_N ($N \neq 2$). A subgroup of this duality symmetry and its action on the moduli has been discussed in [8]. For simplicity we will consider Wilson line modulus with only two gauge indices. Generalization to more than two gauge indices is straightforward. Like G and B, the Wilson line modulus also satisfies consistency requirement. If we represent the Wilson line components, in an orthonormal basis of the gauge lattice, by the matrix

$$A = \begin{pmatrix} A^1{}_1 & A^1{}_2 \\ A^2{}_1 & A^2{}_2 \end{pmatrix}, \tag{25}$$

then consistency condition implies

$$MA = AQ, \tag{26}$$

where M defines the action of the twist on the gauge quantum numbers in the same way Θ defines the action of the twist on the winding and momenta. Using the methods described in the

previous section, we obtain [9]

$$
\begin{aligned}
|\mathbf{P}_R|^2 &= \frac{|in_1T - n_2U_0T - im_1U_0 + m_2 + iu_1Q_1\mathbf{A} - Q_2u_1\mathbf{A}|^2}{\frac{1}{2}(U_0+\bar{U}_0)(T+\bar{T}) - (u_1^2 A\bar{A})}, \\
S &= \frac{1}{2}(Q_1^2 + Q_2^2) + m_1n_1 + m_2n_2.
\end{aligned} \tag{27}
$$

Here Q_1 and Q_2 are the gauge quantum numbers with respect to an orthonormal gauge lattice basis, and the complex moduli T and $\mathbf{A}$ are given in terms of the real components of the Wilson line and the metric by

$$
\begin{aligned}
\mathbf{A} &= 2u_1({A^1}_1 - i{A^2}_1), \\
T &= 2\Big(\sqrt{\det G}(1 + \frac{1}{4}\frac{{A^a}_1 A_{a1}}{G_{11}}) - i(b + \frac{1}{4}\frac{{A^a}_1 A_{a1} G_{12}}{G_{11}} - \frac{1}{4}{A^a}_1 A_{a2})\Big)
\end{aligned} \tag{28}
$$

and $U_0 = u_1 + iu_2$ is the fixed value of the U modulus. Due to the consistency conditions (16) and (26), the following conditions must be satisfied

$$
\begin{aligned}
{A^2}_2 =& u_1{A^1}_1 - u_2{A^2}_1, \\
{A^1}_2 =& -u_2{A^1}_1 - u_1{A^2}_1, \\
G_{12} =& -u_2G_{11}, \\
G_{22} =& |U_0|^2 G_{11}.
\end{aligned} \tag{29}
$$

The moduli T and $\mathbf{A}$ parametrize the moduli space $\frac{SU(2,1)}{SU(2)\otimes U(1)}$. Again in order to identify the duality group in terms of the group $SU(2,1)$, we perform the following change of variables,

$$
\frac{T}{2u_1} = \frac{1-t}{1+t}, \quad \mathbf{A} = 2\frac{\mathcal{A}}{1+t}. \tag{30}
$$

In terms of the new parametrization (30) of the moduli space, $|\mathbf{P}_R|^2$ and S in (27) can be written as

$$
|\mathbf{P}_R|^2 = 2\frac{|Q_c\mathcal{A} - n_ct + m_c|^2}{(1 - t\bar{t} - \mathcal{A}\bar{\mathcal{A}})} = v_c^\dagger \xi_c v_c, \qquad S = v_c^\dagger \mathrm{L} v_c. \tag{31}
$$

where

$$
v_c = \begin{pmatrix} Q_c \\ n_c \\ m_c \end{pmatrix}, \qquad \mathrm{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad Q_c = \frac{1}{\sqrt{2}}(iQ_1 - Q_2),
$$
$$
\xi_c = \frac{2}{1 - t\bar{t} - \mathcal{A}\bar{\mathcal{A}}} \begin{pmatrix} \mathcal{A}\bar{\mathcal{A}} & -t\bar{\mathcal{A}} & \bar{\mathcal{A}} \\ -\bar{t}\mathcal{A} & t\bar{t} & -\bar{t} \\ \mathcal{A} & -t & 1 \end{pmatrix} \tag{32}
$$

and m_c, n_c are as given in (18).

An element of $SU(2,1)$, Γ, is a 3×3 matrix with unit determinant satisfying

$$
\Gamma = \begin{pmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \\ z_7 & z_8 & z_9 \end{pmatrix}, \qquad \Gamma^\dagger \mathrm{L} \Gamma = \mathrm{L}. \tag{33}
$$

The action of Γ on the complex quantum numbers and moduli can be represented by

$$
v_c \rightarrow \Gamma^{-1} v_c, \qquad \xi_c \rightarrow \Gamma^\dagger \xi_c \Gamma. \tag{34}
$$

This gives the following transformations for t and $\mathcal{A}$,

$$
\begin{aligned}
t \rightarrow \tilde{t} &= \frac{z_5 t - z_2 \mathcal{A} - z_8}{z_3 \mathcal{A} + z_9 - z_6 t}, \\
\mathcal{A} \rightarrow \tilde{\mathcal{A}} &= \frac{z_1 \mathcal{A} - z_4 t + z_7}{z_3 \mathcal{A} + z_9 - z_6 t}.
\end{aligned} \tag{35}
$$

Moreover, in addition to (33) there are further constraints on the elements of Γ, these conditions arise from the fact the physical quantum numbers should transform as integers under the duality transformations.

As an illustration, consider the $\mathbf{Z}_3$ orbifold where the internal lattice and the gauge lattice are both given by the root lattice of $SU(3)$ and the twist is defined by the Coxeter action. In

this case, the action of the twist on the winding numbers and quantum gauge numbers is given by

$$Q = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad M = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \tag{36}$$

where an orthonormal basis for the gauge lattice has been chosen. The orthonormal vectors E^a, $a = 1, 2$, are given in terms of the $SU(3)$ root lattice vectors e^I by

$$\begin{aligned} e^1 =& \sqrt{2}E^1, \\ e^2 =& -\frac{1}{\sqrt{2}}E^1 + \sqrt{\frac{3}{2}}E^2. \end{aligned} \tag{37}$$

Substituting (36) in (16) and (26) and solve for the background fields G and A we get

$$\begin{aligned} &G_{12} = -\frac{1}{2}G_{11}, \qquad G_{22} = G_{11}, \\ &A^1{}_2 = -\frac{1}{2}A^1{}_1 - \frac{\sqrt{3}}{2}A^2{}_1, \qquad A^2{}_2 = \frac{\sqrt{3}}{2}A^1{}_1 - \frac{1}{2}A^2{}_1. \end{aligned} \tag{38}$$

The generalization of the above calculations to the $\frac{SU(n,1)}{SU(n)\otimes U(1)}$ cases where one has $n-1$ complex Wilson lines is straightforward. In these cases, the Wilson line and Q_c in (31) will have indices ranging from 1 to $n-1$ and the duality symmetry in terms of $SU(n,1)$ can be derived along the same lines discussed for the case of $SU(2,1)$.

We now discuss the action of duality symmetry on the tree level low-energy Lagrangian. The Kähler potential for the coset $\frac{SU(2,1)}{SU(2)\otimes U(1)}$ coset is given by [8, 12]

$$K = -log(1 - t\bar{t} - \mathcal{A}\bar{\mathcal{A}}).$$

Using the conditions which arise from (33)

$$\begin{aligned} &\bar{z}_1 = z_5 z_9 - z_6 z_8, \qquad \bar{z}_4 = z_3 z_8 - z_2 z_9, \qquad \bar{z}_7 = z_3 z_5 - z_2 z_6, \\ &\bar{z}_2 = z_6 z_7 - z_4 z_9, \qquad \bar{z}_5 = z_1 z_9 - z_3 z_7, \qquad \bar{z}_8 = z_1 z_6 - z_3 z_4, \\ &\bar{z}_3 = z_5 z_7 - z_4 z_8, \qquad \bar{z}_6 = z_1 z_8 - z_2 z_7, \qquad \bar{z}_9 = z_1 z_5 - z_2 z_4, \end{aligned} \tag{39}$$

it can be easily seen that the duality transformations (35) induce a Kähler transformation on

the Kähler potential,

$$K \to K + log(z_3\mathcal{A} + z_9 - z_6 t) + log(\bar{z}_3\bar{\mathcal{A}} + \bar{z}_9 - \bar{z}_6\bar{t}). \qquad (40)$$

As was mentioned in the introduction, ignoring gauge terms, the low-energy effective Lagrangian is described in terms of the function $\mathcal{G}$ defined in eqn (1). Therefore for the low-energy theory to be invariant under the duality transformation (35) associated with the coset $\frac{SU(2,1)}{SU(2)\otimes U(1)}$, the superpotential must transform as (up to a $\mathcal{A}$, t-independent phase)

$$W \to W(z_3\mathcal{A} + z_9 - z_6 t)^{-1}. \qquad (41)$$

Eq. (41) provides a non-trivial constraint on the form of any possible non-perturbative superpotential. The duality transformations for the various fields in the low-energy effective action can be determined from the form of their associated vertex operators [18]. It would be interesting to investigate the duality transformations of the twist fields and their interactions and verify that (41) holds for the Yukawa couplings in the twisted sectors.

To summarize, we have analyzed the duality structure of the coset spaces $\frac{SU(n,1)}{SU(n)\otimes U(1)}$. These spaces describe the moduli spaces of a two-dimensional torus in a factorizable orbifold where the twist has a complex eigenvalue and are parametrized by a complex Kähler moduli $\mathbf{T}$ and $n-1$ complex Wilson moduli $\mathbf{A}$. Also, the duality transformations of the basic physical parameters in the low-energy effective action are discussed.

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